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Current Fluctuations in the One-Dimensional Symmetric Exclusion Process with Open Boundaries

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We calculate the first four cumulants of the integrated current of the one-dimensional symmetric simple exclusion process of N sites with open boundary conditions. For large system size N, the generating function of the integrated current depends on the densities ρ_a and ρ_b of the two reservoirs and on the fugacity z, the parameter conjugated to the integrated current, through a single parameter. Based on our expressions for these first four cumulants, we make a conjecture which leads to a prediction for all the higher cumulants. In the case $\rho_a = 1$ and $\rho_b = 0$, our conjecture gives the same universal distribution as the one obtained by Lee, Levitov, and Yakovets for one-dimensional quantum conductors in the metallic regime.

KEY WORDS: Large deviations; symmetric simple exclusion process; open system; stationary nonequilibrium state; current fluctuations; ruin problems; diffusive medium; full counting statistics; shot noise.

1. INTRODUCTION

The study of the current through a system in contact with two reservoirs at unequal chemical potentials or at unequal temperatures is one of the most studied aspects of the theory of non-equilibrium systems.[1, 2]

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For the last decade, there has been an increasing interest in the study of the fluctuations of the current of quantum particles (fermions) through a disordered wire. It is now well established that the quantum statistics of the particles determines the distribution of the fluctuations of the current, and that in the metallic regime,[3,4] this distribution[5–7] is universal. More recent works have shown that the main property of the quantum nature of the particles which was responsible for these universal fluctuations is the Pauli exclusion principle.[8–12]

Here we consider the symmetric simple exclusion process SSEP[13–17] which is a stochastic model of classical particles with hard core interactions (and without inertia) which diffuse on a finite chain with open boundary conditions. The chain is in contact at its two ends with two reservoirs of particles at unequal densities.[18–22] The combined effects of the stochastic injection and removal of particles at the two boundaries and of the diffusive nature of the hard core particles produce a fluctuating current. We calculate the first four cumulants of the integrated current for a long chain. Based on our results for these four cumulants, we give a conjecture for all the higher cumulants and for the whole distribution of the current fluctuations.

The fluctuations of the current in exclusion processes is also a subject with a long history.[23–27] Most of the known results obtained so far concern infinite geometries[23,24,27] or systems with periodic boundary conditions[26,28–30] (see ref. 31 for the variance of the integrated current of the asymmetric simple exclusion process with open boundaries).

Our paper is organised as follows: in Section 2 we define the model and we summarize our results. In Section 3, we show how the first two cumulants can be calculated from the steady state properties. In Section 4, we write a hierarchy (see also Appendix C) for the correlation functions on which our approach is based. In Sections 5 and 6 we solve this hierarchy, in a low density expansion, where at each order the hierarchy can be truncated. Appendix A gives a derivation of the Gallavotti–Cohen relation[32,33] for the SSEP with open boundaries. Appendix B points out the analogy with multi-particle ruin problems.

2. DEFINITION OF THE MODEL AND MAIN RESULTS

2.1. The Symmetric Exclusion Process with Open Boundaries

In the one-dimensional symmetric simple exclusion process, each site i (with $1 \leqslant i \leqslant N$) of a one-dimensional lattice of N sites is either occupied by a single particle or empty. A configuration $\mathscr{C}$ at time t is therefore fully determined by N binary variables $\tau_i(t)$, the occupation numbers of the

N sites ($\tau_i(t)=1$ if site i is occupied and $\tau_i(t)=0$ if site i is empty). In the bulk, each particle independently attempts to jump to its right neighboring site, and to its left neighboring site, in each case at rate 1. It succeeds if the target site is empty; otherwise nothing happens (this means that during time t and time $t+dt$ with $0<dt\ll 1$, a particle at site i jumps to site $i-1$ with probability $(1-\tau_{i-1})\,dt$, to site $i+1$ with probability $(1-\tau_{i+1})\,dt$ and does not move with probability $1-(2-\tau_{i-1}-\tau_{i+1})\,dt$). At the left boundary particles are injected at site 1 at rate α and removed from site 1 at rate γ. Similarly at the right boundary, particles are removed from site N at rate β and injected at site N at rate δ.

For general values of α, β, γ, δ, a current of particles flows through the system and we want to study the fluctuations of this current. To do so, we denote by $Q(t)$ the number of particles which have moved from the left reservoir into the system during time t (so $Q(t)$ is the number of particles which have jumped into the system at site 1 minus the number of particles which have left the system from site 1). We want to calculate the distribution of the total charge $Q(t)$ during a long time t.

For finite N the system has 2^N internal configurations $\mathscr{C}$ (each site can be either occupied by a particle or empty). Let $p_t(\mathscr{C})$ be the probability of finding the system in configuration $\mathscr{C}$ at time t. As the dynamics is a Markov process, the evolution of the probability $p_t(\mathscr{C})$ of finding the system in configuration $\mathscr{C}$ at time t can be written as

$$\frac{dp_t(\mathscr{C})}{dt}=\sum_{\mathscr{C}'}\left[W_1(\mathscr{C},\mathscr{C}')+W_0(\mathscr{C},\mathscr{C}')+W_{-1}(\mathscr{C},\mathscr{C}')\right]p_t(\mathscr{C}') \tag{2.1}$$

where we have decomposed the Markov matrix into three parts, depending on whether when the system jumps from configuration $\mathscr{C}'$ to configuration $\mathscr{C}$, $Q(t)$ increases by 1, 0, or -1 (the matrix W_0 contains all the diagonal terms which are all negative as well as all the non-diagonal elements corresponding to moves which do not take place at the left boundary, i.e., do not change $Q(t)$). One way to determine the distribution of $Q(t)$ is to calculate its generating function $\langle z^{Q(t)}\rangle$.

If we define $P_t(\mathscr{C},Q)$ the probability that the system is in configuration $\mathscr{C}$ at time t and that $Q(t)=Q$, one has

$$\begin{aligned}\frac{dP_t(\mathscr{C},Q)}{dt}=\sum_{\mathscr{C}'}&W_1(\mathscr{C},\mathscr{C}')\,P_t(\mathscr{C}',Q-1)+W_0(\mathscr{C},\mathscr{C}')\,P_t(\mathscr{C}',Q)\\&+W_{-1}(\mathscr{C},\mathscr{C}')\,P_t(\mathscr{C}',Q+1).\end{aligned} \tag{2.2}$$

Then the generating functions $\mathcal{P}_t(\mathcal{C}, z)$ defined by

$$\mathcal{P}_t(\mathcal{C}, z) = \sum_{Q=-\infty}^{\infty} P_t(\mathcal{C}, Q)\, z^Q \tag{2.3}$$

satisfy

$$\frac{d\mathcal{P}_t(\mathcal{C}, z)}{dt} = \sum_{\mathcal{C}'} \left[zW_1(\mathcal{C}, \mathcal{C}') + W_0(\mathcal{C}, \mathcal{C}') + \frac{1}{z} W_{-1}(\mathcal{C}, \mathcal{C}') \right] \mathcal{P}_t(\mathcal{C}', z). \tag{2.4}$$

If we introduce the matrix M_z defined by

$$M_z(\mathcal{C}, \mathcal{C}') = zW_1(\mathcal{C}, \mathcal{C}') + W_0(\mathcal{C}, \mathcal{C}') + \frac{1}{z} W_{-1}(\mathcal{C}, \mathcal{C}') \tag{2.5}$$

it is clear from (2.4) that in the long time limit

$$\langle z^{Q(t)} \rangle = \sum_{\mathcal{C}} \mathcal{P}_t(\mathcal{C}, z) \sim e^{\mu t} \tag{2.6}$$

where μ is the largest eigenvalue of the matrix M_z. So this largest eigenvalue μ fully determines the distribution of $Q(t)$ in the long time limit.[26]

2.2. Symmetries of μ

In principle, μ depends on six parameters: the input rates α, β, γ, δ at the two boundaries, the fugacity z and the number of sites N. There are three symmetries in the system that leave μ unchanged:

1. *The left-right symmetry*: if we exchange the roles of α, γ and δ, β, this has the effect of exchanging the roles of the left and of the right boundaries, and so the statistical properties of $Q(t)$ are replaced by those of $-Q(t)$. Therefore μ should satisfy

$$\mu(\alpha, \gamma, \delta, \beta, z, N) = \mu\left(\delta, \beta, \alpha, \gamma, \frac{1}{z}, N\right). \tag{2.7}$$

2. *The particle-hole symmetry*: instead of counting the number of particles $Q(t)$ entering at the left boundary, one can as well count the number $-Q(t)$ of holes entering at the left boundary. Now the holes are injected at rate γ and removed at rate α at the left boundary and they are injected at rate β and removed at rate δ at the right boundary. They also

jump with the same exclusion rules as the particles in the bulk. Therefore, this symmetry implies that

$$\mu(\alpha, \gamma, \delta, \beta, z, N) = \mu\left(\gamma, \alpha, \beta, \delta, \frac{1}{z}, N\right). \tag{2.8}$$

3. *The Gallavotti–Cohen symmetry*: the rates $\alpha, \beta, \gamma, \delta$ represent the transfer of particles between the system and reservoirs at densities $\rho_a = \frac{\alpha}{\alpha+\gamma}$ and $\rho_b = \frac{\delta}{\beta+\delta}$ at the two boundaries (sites 1 and N). When $\rho_a = \rho_b$, the system is in equilibrium and the dynamics satisfy detailed balance with respect to a Bernoulli measure[20] at density $\rho = \rho_a = \rho_b$. One can always think that the case $\rho_a \neq \rho_b$ represents the effect of an external field which enhances the flux of particles from one reservoir into the system, a situation for which (as explained in the Appendix A) the Gallavotti–Cohen relation holds.[32, 33] This implies that

$$\mu(\alpha, \gamma, \delta, \beta, z, N) = \mu\left(\alpha, \gamma, \delta, \beta, \frac{\gamma\delta}{\alpha\beta z}, N\right). \tag{2.9}$$

2.3. Main Results

When N is large, one finds, at least pertubatively in powers of α and γ (see Sections 5 and 6) that μ depends only on the densities ρ_a and ρ_b of the left and right reservoirs

$$\rho_a = \frac{\alpha}{\alpha+\gamma}; \qquad \rho_b = \frac{\delta}{\beta+\delta} \tag{2.10}$$

instead of the four parameters α, β, γ, and δ.

The three symmetries (2.7)–(2.9) then become

$$\mu(\rho_a, \rho_b, z, N) = \mu\left(\rho_b, \rho_a, \frac{1}{z}, N\right) \tag{2.11}$$

$$\mu(\rho_a, \rho_b, z, N) = \mu\left(1-\rho_a, 1-\rho_b, \frac{1}{z}, N\right) \tag{2.12}$$

$$\mu(\rho_a, \rho_b, z, N) = \mu\left(\rho_a, \rho_b, \frac{\rho_b(1-\rho_a)}{z\rho_a(1-\rho_b)}, N\right). \tag{2.13}$$

It is also a fact observed in the perturbation theory to arbitrary order (see Sections 5 and 6) that, for large N, μ is proportional to $1/N$ times a function of a single variable ω defined by

$$\omega = \frac{(z-1)(\rho_a z - \rho_b - \rho_a \rho_b (z-1))}{z} \tag{2.14}$$

and the result of our expansion in powers of ω of Section 6 is that

$$\mu = \frac{1}{N} R(\omega) + O\left(\frac{1}{N^2}\right) \tag{2.15}$$

where

$$R(\omega) = \omega - \frac{\omega^2}{3} + \frac{8\omega^3}{45} - \frac{4\omega^4}{35} + O(\omega^5). \tag{2.16}$$

The symmetries (2.11)–(2.13) leave ω given by (2.14) unchanged so that μ given by (2.14), (2.15) satisfies automatically these symmetries. From (2.16) one can easily obtain the large N expression of the first four cumulants of $Q(t)$

$$\lim_{t\to\infty} \frac{\langle Q(t)\rangle}{t} \simeq \frac{1}{N}[\rho_a - \rho_b] \tag{2.17}$$

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle_c}{t} \simeq \frac{1}{N}\left[\rho_a + \rho_b - \frac{2(\rho_a^2 + \rho_a\rho_b + \rho_b^2)}{3}\right] \tag{2.18}$$

$$\lim_{t\to\infty} \frac{\langle Q^3(t)\rangle_c}{t} \simeq \frac{1}{N}(\rho_a - \rho_b)\left[1 - 2(\rho_a+\rho_b) + \frac{16\rho_a^2 + 28\rho_a\rho_b + 16\rho_b^2}{15}\right] \tag{2.19}$$

$$\begin{aligned}\lim_{t\to\infty} \frac{\langle Q^4(t)\rangle_c}{t} \simeq \frac{1}{N}\Bigg[&\rho_a + \rho_b - \frac{2(7\rho_a^2 + \rho_a\rho_b + 7\rho_b^2)}{3} \\ &+ \frac{32\rho_a^3 + 8\rho_a^2\rho_b + 8\rho_a\rho_b^2 + 32\rho_b^3}{5} \\ &- \frac{96\rho_a^4 + 64\rho_a^3\rho_b - 40\rho_a^2\rho_b^2 + 64\rho_a\rho_b^3 + 96\rho_b^4}{35}\Bigg].\end{aligned} \tag{2.20}$$

One can notice that the nth cumulant (at least for $n \leqslant 4$) is a polynomial of degree n in ρ_a, ρ_b. We will comment on this at the end of Section 5.

2.4. Two Particular Cases

Let us examine these expressions in two particular cases. First when

$$\rho_a = 1 \qquad \text{and} \qquad \rho_b = 0 \tag{2.21}$$

one sees that (2.17)–(2.20) give in the long time limit

$$\frac{\langle Q(t)\rangle}{t} \to \frac{1}{N} \tag{2.22}$$

$$\frac{\langle Q(t)^2\rangle_c}{t} \to \frac{1}{3N} \tag{2.23}$$

$$\frac{\langle Q(t)^3\rangle_c}{t} \to \frac{1}{15N} \tag{2.24}$$

$$\frac{\langle Q(t)^4\rangle_c}{t} \to \frac{-1}{105N}. \tag{2.25}$$

These numbers coincide with those found for quantum conductors with many channels in the metallic regime[5] and for quasi-classical conductors analysed by a Boltzmann–Langevin approach.[10]

Another particular case of interest is when the two reservoirs are at the same density ρ

$$\rho_a = \rho_b = \rho. \tag{2.26}$$

All the odd cumulants vanish and (2.18), (2.20) become

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle_c}{t} \simeq \frac{1}{N} 2\rho(1-\rho) \tag{2.27}$$

$$\lim_{t\to\infty} \frac{\langle Q^4(t)\rangle_c}{t} \simeq \frac{1}{N} 2\rho(1-\rho)(1-2\rho)^2. \tag{2.28}$$

2.5. Conjecture

We see in (2.28) that the fourth cumulant vanishes when $\rho_a = \rho_b = 1/2$. We conjecture that in this particular case, $\rho_a = \rho_b = 1/2$, all the higher cumulants vanish (i.e., the distribution is Gaussian) so that μ is given in this case by

$$\mu = \frac{1}{N} \frac{(\log z)^2}{4} + O\left(\frac{1}{N^2}\right).$$

This conjecture (see (2.14) and (2.15)) fully determines the function $R(\omega)$

$$R(\omega) = [\log(\sqrt{1+\omega}+\sqrt{\omega})]^2 \tag{2.29}$$

and therefore μ, using (2.15), (2.29) for arbitrary ρ_a, ρ_b, and z.

Expression (2.29) needs to be modified when ω becomes negative. We will also conjecture that for $\omega < 0$, (2.29) is replaced by its analytic continuation

$$R(\omega) = -[\sin^{-1}(\sqrt{-\omega})]^2. \tag{2.30}$$

Looking again at the first case we analyzed (2.21), we get that not only the first four cumulants (2.22)–(2.25) are the same as those of the distribution first obtained by Lee *et al.*,[5] but all the higher cumulants are the same

$$\frac{\langle Q(t)^5\rangle_c}{t} \to \frac{-1}{105N}$$

$$\frac{\langle Q(t)^6\rangle_c}{t} \to \frac{1}{231N}$$

$$\frac{\langle Q(t)^7\rangle_c}{t} \to \frac{27}{5005N}$$

$$\frac{\langle Q(t)^8\rangle_c}{t} \to \frac{-3}{715N}.$$

In the equilibrium case ($\rho_a = \rho_b = \rho$) too, this conjecture determines all the cumulants higher than (2.27), (2.28)

$$\frac{\langle Q(t)^6\rangle_c}{t} \to 2\rho(1-\rho)(2\rho-1)^2\,(1-16\rho+16\rho^2)$$

$$\frac{\langle Q(t)^8\rangle_c}{t} \to 2\rho(1-\rho)(2\rho-1)^2\,(1-80\rho+656\rho^2-1152\rho^3+576\rho^4).$$

Our conjecture for the distribution of $Q(t)$ for arbitrary ρ_a and ρ_b coincides with the distribution found in a multi-channel quantum picture[7] (with a small discrepancy with the distribution proposed in ref. 6).

2.6. The Large Deviation Function

The knowledge of the large N behaviour of μ gives some information on the large deviation function $F_N(q)$. This large deviation function $F_N(q)$ is defined by

$$\text{Probability}\left(\frac{Q(t)}{t} \simeq q\right) \sim \exp[tF_N(q)] \tag{2.31}$$

or for a more mathematical definition

$$\lim_{t\to\infty} \frac{1}{t} \log[\text{Probability}(tq \leqslant Q(t) < tq+1)] = F_N(q). \tag{2.32}$$

If we knew $\mu(\alpha, \beta, \delta, \gamma, z, N)$, one would determine $F_N(q)$ in a parametric form by varying z

$$q = z\frac{\partial\mu}{\partial z}$$

$$F_N(q) = \mu - q\log z.$$

As here we know only μ only for large N (see (2.15)), we cannot get the full large deviation function $F_N(q)$ for arbitrary N but we can say that in the large N limit,

$$\lim_{N\to\infty} NF_N\left(\frac{q}{N}\right) = G(q) \tag{2.33}$$

where the function $G(q)$ can be constructed from $R(\omega)$ in a parametric form by varying z

$$q = z\left(\frac{d\omega}{dz}\right)\left(\frac{dR(\omega)}{d\omega}\right) \tag{2.34}$$

$$G(q) = R(\omega) - q\log z. \tag{2.35}$$

This means that for large N we know $F_N(q)$ only for deviations q of order $1/N$. Figure 1 shows $G(q)$ versus q for two choices of ρ_a and ρ_b (the case $\rho_a = 1$ and $\rho_b = 0$ and the case $\rho_a = \rho_b = 0.25$).

3. THE AVERAGE CURRENT AND ITS VARIANCE

In this section we show that the expected value and the variance of the integrated current $Q(t)$ can be calculated easily by using the conservation rules.

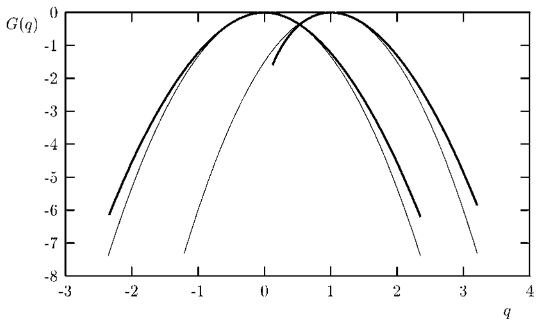

Fig. 1. The rescaled large deviation functions $G(q)$ versus q in the cases $\rho_a = \rho_b = 1/4$ (left thick curve) and $\rho_a = 1$ and $\rho_b = 0$ (right thick curve). The thin lines represent for comparison the Gaussians with the same two moments.

Let us define $Y_i(t)$ the integrated current between sites i and $i+1$ during the time interval 0, t (so $Y_i(t)$ is the total number of particles which have jumped from i to $i+1$ minus the number of particles which have jumped from $i+1$ to i during time t). Similarly let us define $Y_0(t)$ the integrated current from the left reservoir to site 1 and $Y_N(t)$ the integrated current from site N to the right reservoir. Note that $Y_0(t)$ and $Q(t)$ have exactly the same definition and therefore

$$Q(t) = Y_0(t).$$

The conservation of the number of particles implies that

$$Y_i(t) = Y_{i-1}(t) + \tau_i(0) - \tau_i(t). \tag{3.1}$$

The difference between $Y_i(t)$ and $Y_j(t)$ remains bounded ((3.1) implies that $|Y_i(t) - Y_{i-1}(t)| \leqslant 1$ and $|Y_i(t) - Y_j(t)| \leqslant 1\ |j-i|$). Therefore in the long time limit the cumulants of $Y_i(t)$ do not depend on i.

$$\lim_{t\to\infty} \frac{\log \langle z^{Q(t)} \rangle}{t} = \lim_{t\to\infty} \frac{\log \langle z^{Y_i(t)} \rangle}{t} = \lim_{t\to\infty} \frac{\log \langle z^{Y_j(t)} \rangle}{t}. \tag{3.2}$$

The very definition of the dynamics in Section 2 means that during each time interval $dt \ll 1$,

$$\begin{aligned} Y_0(t+dt) = Y_0(t) \qquad & \text{with probability} \quad 1 - \alpha(1-\tau_1)\, dt - \gamma\tau_1\, dt \\ Y_0(t)+1 \qquad & \text{with probability} \quad \alpha(1-\tau_1)\, dt \\ Y_0(t)-1 \qquad & \text{with probability} \quad \gamma\tau_1\, dt. \end{aligned}$$

From this evolution one can deduce the following time evolution for the moments of $Y_0(t)$:

$$\frac{d\langle Y_0(t)\rangle}{dt} = \alpha - (\alpha+\gamma)\langle \tau_1 \rangle \tag{3.3}$$

$$\frac{d\langle Y_0(t)^2\rangle}{dt} = 2\alpha\langle Y_0(t)\rangle - 2(\alpha+\gamma)\langle Y_0(t)\,\tau_1\rangle + \alpha + (\gamma-\alpha)\langle \tau_1\rangle \tag{3.4}$$

and more generally

$$\frac{d\langle Y_0(t)^k\rangle}{dt} = \alpha\langle [(Y_0(t)+1)^k - Y_0(t)^k](1-\tau_1)\rangle + \gamma\langle [(Y_0(t)-1)^k - Y_0(t)^k]\,\tau_1\rangle.$$

From (3.3), (3.4) we obtain

$$\frac{d}{dt}\langle Y_0(t)^2\rangle - \langle Y_0(t)\rangle^2$$
$$= -2(\alpha+\gamma)[\langle Y_0(t)\,\tau_1\rangle - \langle Y_0(t)\rangle\langle \tau_1\rangle] + \alpha + (\gamma-\alpha)\langle \tau_1\rangle. \tag{3.5}$$

Similarly starting from the dynamics of the integrated current $Y_i(t)$ through the bond $i, i+1$ or of the integrated current $Y_N(t)$ between site N and the right reservoir one can get

$$\frac{d\langle Y_i(t)\rangle}{dt} = \langle \tau_i\rangle - \langle \tau_{i+1}\rangle \tag{3.6}$$

$$\frac{d}{dt}\langle Y_i(t)^2\rangle - \langle Y_i(t)\rangle^2 = 2[\langle Y_i(t)(\tau_i - \tau_{i+1})\rangle - \langle Y_i(t)\rangle\langle \tau_i - \tau_{i+1}\rangle]$$
$$+\langle \tau_i + \tau_{i+1} - 2\tau_i\tau_{i+1}\rangle \tag{3.7}$$

and

$$\frac{d\langle Y_N(t)\rangle}{dt} = (\beta+\delta)\langle \tau_N\rangle - \delta \tag{3.8}$$

$$\frac{d}{dt}\langle Y_N(t)^2\rangle - \langle Y_N(t)\rangle^2 = 2(\beta+\delta)[\langle Y_N(t)\,\tau_N\rangle - \langle Y_N(t)\rangle\langle \tau_N\rangle]$$
$$+\delta + (\beta-\delta)\langle \tau_N\rangle. \tag{3.9}$$

3.1. The Current

If we define the parameters a, b, ρ_a, ρ_b as in refs. 19 and 20, and (2.10)

$$a = \frac{1}{\alpha+\gamma}; \qquad b = \frac{1}{\beta+\delta}$$
$$\rho_a = \frac{\alpha}{\alpha+\gamma}; \qquad \rho_b = \frac{\delta}{\beta+\delta} \tag{3.10}$$

one obtains by combining (3.3), (3.6), (3.8) that

$$a\frac{d\langle Y_0\rangle}{dt} + b\frac{d\langle Y_N\rangle}{dt} + \sum_{i=1}^{N-1} \frac{d\langle Y_i\rangle}{dt} = \rho_a - \rho_b.$$

We know (3.1), (3.2) that in the steady state $d\langle Y_i\rangle/dt$ does not depend on i. Therefore, one obtains that way the steady state current

$$\frac{d\langle Q\rangle}{dt} = \frac{d\langle Y_i\rangle}{dt} = \frac{\rho_a - \rho_b}{N+a+b-1} \tag{3.11}$$

which gives (2.17) for large N.

3.2. The Variance

Similarly adding (3.5), (3.7), (3.9) and using the fact that in the steady state $\frac{d\langle Y_i^2\rangle - \langle Y_i\rangle^2}{dt}$ does not depend on i one gets

$$(a+b+N-1)\frac{d\langle Y^2\rangle - \langle Y\rangle^2}{dt}$$
$$= 2\sum_{i=1}^{N} \langle Y_i\tau_i\rangle - \langle Y_i\rangle\langle\tau_i\rangle - \langle Y_{i-1}\tau_i\rangle + \langle Y_{i-1}\rangle\langle\tau_i\rangle$$
$$+\rho_a + \rho_b - 2\rho_a\langle\tau_1\rangle - 2\rho_b\langle\tau_N\rangle + 2\sum_{i=1}^{N}\langle\tau_i\rangle - 2\sum_{i=1}^{N-1}\langle\tau_i\tau_{i+1}\rangle \tag{3.12}$$

and using (3.1) one obtains (using that $\langle\tau_i(0)\,\tau_i(t)\rangle \to \langle\tau_i(0)\rangle\langle\tau_i\rangle$ in the long time limit)

$$(a+b+N-1)\frac{d\langle Y^2\rangle - \langle Y\rangle^2}{dt}$$
$$= \rho_a + \rho_b - 2\rho_a\langle\tau_1\rangle - 2\rho_b\langle\tau_N\rangle + 2\sum_{i=1}^{N}\langle\tau_i\rangle^2 - 2\sum_{i=1}^{n-1}\langle\tau_i\tau_{i+1}\rangle. \tag{3.13}$$

All the steady state correlations can be calculated exactly,[20, 34] in particular

$$\langle \tau_i \rangle = \rho_b + \frac{N-i+b}{N+a+b-1}(\rho_a - \rho_b) = \frac{\rho_a(N+b-i)+\rho_b(i-1+a)}{N+a+b-1}$$

and for $i<j$

$$\langle \tau_i \tau_j \rangle - \langle \tau_i \rangle \langle \tau_j \rangle = -(\rho_b - \rho_a)^2 \frac{(a+i-1)(b+N-j)}{(N+a+b-1)^2\,(N+a+b-2)}$$

so that (3.13) becomes

$$\begin{aligned}
&\frac{d[\langle Q^2 \rangle - \langle Q \rangle^2]}{dt} \\
&\quad = \frac{d[\langle Y_i^2 \rangle - \langle Y_i \rangle^2]}{dt} \\
&\quad = \frac{1}{N_1}(\rho_a + \rho_b - 2\rho_a \rho_b) \\
&\qquad + \frac{a(a-1)(2a-1)+b(b-1)(2b-1)-N_1(N_1-1)(2N_1-1)}{3N_1^3(N_1-1)}(\rho_a - \rho_b)^2
\end{aligned} \tag{3.14}$$

where $N_1 = N+a+b-1$. In the large N limit, one obtains (2.18).

4. A HIERARCHY OF EQUATIONS FOR THE CORRELATION FUNCTIONS

In the long time limit, the vector $\mathscr{P}_t(\mathscr{C}, z)$ in (2.4) becomes an eigenvector of the matrix M_z defined in (2.5)

$$\mathscr{P}_t(\mathscr{C}, z) \sim e^{\mu t} \psi_\mu(\mathscr{C})$$

where $\psi_\mu(\mathscr{C})$ satisfies

$$\mu \psi_\mu(\mathscr{C}) = \sum_{\mathscr{C}'} M_z(\mathscr{C}, \mathscr{C}')\, \psi_\mu(\mathscr{C}'). \tag{4.1}$$

From (4.1), one can build a hierarchy of equations which, as we shall see it in the next Section 5, can be truncated either when one expands in powers of $z-1$ to obtain the first cumulants or when the densities ρ_a and ρ_b in the reservoirs are small.

Let us define the following correlation functions:

for $1 \leqslant i \leqslant N$

$$T_i = \sum_{\mathscr{C}} \psi_\mu(\mathscr{C})\, \tau_i(\mathscr{C}) \tag{4.2}$$

for $1 \leqslant i < j \leqslant N$

$$U_{i,j} = \sum_{\mathscr{C}} \psi_\mu(\mathscr{C})\, \tau_i(\mathscr{C})\, \tau_j(\mathscr{C}) \tag{4.3}$$

for $1 \leqslant i < j < k \leqslant N$

$$V_{i,j,k} = \sum_{\mathscr{C}} \psi_\mu(\mathscr{C})\, \tau_i(\mathscr{C})\, \tau_j(\mathscr{C})\, \tau_k(\mathscr{C}) \tag{4.4}$$

and so on, with the convention that

$$\sum_{\mathscr{C}} \psi_\mu(\mathscr{C}) = 1. \tag{4.5}$$

Inserting these definitions into (4.1), one obtains a hierarchy of equations for the one-point functions T_i, the two point functions $U_{i,j}$, and so on. By summing (4.1) over all $\mathscr{C}$, one obtains

$$\mu = \alpha(z-1) + \left(\gamma \frac{1}{z} - \alpha z + \alpha - \gamma\right) T_1. \tag{4.6}$$

By multiplying (4.1) by $\tau_i(\mathscr{C})$ and summing over $\mathscr{C}$, one gets

$$\mu T_i = \alpha(z-1)\, T_i + \left(\gamma \frac{1}{z} - \alpha z + \alpha - \gamma\right) U_{1,i} + T_{i-1} + T_{i+1} - 2T_i. \tag{4.7}$$

At the two boundaries (4.7) is modified

$$\mu T_1 = \alpha z - (\alpha z + \gamma)\, T_1 + T_2 - T_1 \tag{4.8}$$

$$\mu T_N = \alpha(z-1)\, T_N + \left(\gamma \frac{1}{z} - \alpha z + \alpha - \gamma\right) U_{1,N} + T_{N-1} - (1+\beta+\delta)\, T_N + \delta. \tag{4.9}$$

In fact (4.8) and (4.9) (which are the boundary versions of (4.7)) reduce to (4.7) provided that we require that T_0, T_{N+1}, and $U_{1,1}$ (for non-physical values of the parameters) satisfy

$$\alpha(z-1)\,T_1+\left(\gamma\frac{1}{z}-\alpha z+\alpha-\gamma\right)U_{1,1}+T_0-T_1=\alpha z-(\alpha z+\gamma)\,T_1 \tag{4.10}$$

$$\delta-(\beta+\delta)\,T_N=T_{N+1}-T_N. \tag{4.11}$$

Similarly by multiplying by $\tau_i(\mathscr{C})\,\tau_j(\mathscr{C})$ one gets

$$\begin{aligned}\mu U_{i,j}=\alpha(z-1)\,U_{i,j}&+\left(\gamma\frac{1}{z}-\alpha z+\alpha-\gamma\right)V_{1,i,j}\\&+U_{i-1,j}+U_{i+1,j}+U_{i,j-1}+U_{i,j+1}-4U_{i,j}\end{aligned} \tag{4.12}$$

the boundary conditions and the case $j=i+1$ being automatically satisfied provided that the extensions of $U_{0,i}$, $U_{i,i}$, $U_{i,N+1}$, $V_{1,1,i}$ to non-physical values satisfy

$$\alpha(z-1)\,U_{1,i}+\left(\gamma\frac{1}{z}-\alpha z+\alpha-\gamma\right)V_{1,1,i}+U_{0,i}-U_{1,i}=\alpha zT_i-(\alpha z+\gamma)\,U_{1,i} \tag{4.13}$$

$$\delta T_i-(\beta+\delta)\,U_{i,N}=U_{i,N+1}-U_{i,N} \tag{4.14}$$

$$U_{i,i}+U_{i+1,i+1}=2U_{i,i+1}. \tag{4.15}$$

(Note that definitions such as (4.3) do not tell us what $U_{i,i}$ is as $U_{i,j}$ is only defined for $j>i$. In general the values one has to choose for non-physical values of the parameters in order to satisfy the boundary conditions (4.13)–(4.15) are different from what one could obtain by simply putting $j=i$ in the definition: in particular $U_{i,i}\neq T_i$. In this whole paper the $U_{i,j}$ we calculate are always polynomials in i and j and the unphysical values such as $U_{i,i}$ are simply obtained by taking $j=i$ in the polynomial $U_{i,j}$.)

All the relations for the higher correlation functions can be generated along the same steps. One way of writing all these relations is to introduce the generating function

$$\Phi(a_1,\dots,a_L;z)=\left\langle z^{Q(t)}\exp\left[\sum_i a_i\tau_i(t)\right]\right\rangle$$

(where $Q(t)$, as above, is the total number of particles transferred from the left reservoir to site 1 during time t). For large t one expects that

$$\Phi(a_1,\dots,a_L)\sim e^{\mu t}\phi(a_1,\dots,a_L)$$

where ϕ satisfies

$$\mu\phi = \Bigg[\sum_{i=1}^{L-1} (e^{a_{i+1}-a_i} - 1) \left(\frac{\partial}{\partial a_i} - \frac{\partial^2}{\partial a_i\, \partial a_{i+1}} \right)$$

$$+ (e^{a_i - a_{i+1}} - 1) \left(\frac{\partial}{\partial a_{i+1}} - \frac{\partial^2}{\partial a_i\, \partial a_{i+1}} \right)$$

$$+ \alpha(z e^{a_1} - 1) \left(1 - \frac{\partial}{\partial a_1} \right) + \gamma \left(\frac{e^{-a_1}}{z} - 1 \right) \frac{\partial}{\partial a_1}$$

$$+ \delta(e^{a_L} - 1) \left(1 - \frac{\partial}{\partial a_L} \right) + \beta(e^{-a_L} - 1) \frac{\partial}{\partial a_L} \Bigg] \phi. \tag{4.16}$$

Expanding (4.16) in powers of the a_i allows one to recover all the above relations between the correlation functions (4.6), (4.7),..., and to generate the equations satisfied by the higher correlations. The first levels of the hierarchy are summarized in Appendix C.

5. THE LOW DENSITY EXPANSION

When the densities ρ_a and ρ_b of the reservoirs are small the n-point function is of order n to leading order in ρ_a and ρ_b. To calculate μ to order n in ρ_a and ρ_b, one can truncate the hierarchy by neglecting all the m-point correlation functions for $m > n$.

A priori the truncated hierarchy remains a problem hard to solve. However we noticed that *the solutions T_i, $U_{i,j}$,... of the truncated hierarchy are always polynomials in the coordinates i, j,...* (see the Appendix B on the analogy with a multi-particle ruin problem). For example if one tries to expand to order 3 in ρ_a and ρ_b (or in α and γ) one finds that T_i is a polynomial of degree 5 in i, $U_{i,j}$ of degree 4 in i, j and $V_{i,j,k}$ a polynomial of degree 3 in i, j, k (in fact $V_{i,j,k}$ is linear in each of the three coordinates). So to solve the truncated hierarchy, we introduced arbitrary parameters (the coefficients of all the polynomials in i, i, j, i, j, k,...) and the equations of the hierarchy give us a finite set of linear equations to solve for these parameters.

We used Mathematica to solve these linear equations. The expressions become quickly complicated for general a, b, ρ_a, ρ_b. The general expression (C.14) of μ at order 3 in ρ_a and ρ_b is given in the Appendix C. We give here the result obtained that way for μ to order 3 in ρ_a, ρ_b when $a = b = 1$

$$\mu = \frac{(z-1)(\rho_a z - \rho_b)}{z(N+1)} - \frac{(z-1)^2 \left(\rho_b^2 + 4\rho_a\rho_b z + \rho_a^2 z^2 + 2N(\rho_b^2 + \rho_a\rho_b z + \rho_a^2 z^2)\right)}{6z^2(N+1)^2} + \frac{(z-1)^3 (2N+1)(\rho_a z - \rho_b)(3\rho_b^2 + 9\rho_a\rho_b z + 3\rho_a^2 z^2 + N(4\rho_b^2 + 7\rho_a\rho_b z + 4\rho_a^2 z^2))}{45z^3(N+1)^3}. \tag{5.1}$$

For large N the expression (C.14) of μ gets much simpler: to leading order in N, the results do not depend anymore on the two parameters a and b and one gets

$$\mu = \frac{1}{N}\left[\frac{(\rho_a z - \rho_b)(z-1)}{z} - \frac{(\rho_a^2 z^2 + \rho_a\rho_b z + \rho_b^2)(z-1)^2}{3z^2} + \frac{2(\rho_a z - \rho_b)(4\rho_a^2 z^2 + 7\rho_a\rho_b z + \rho_b^2)(z-1)^3}{45z^3} + O(\rho^4)\right]. \tag{5.2}$$

So in this large N regime, μ is proportional to $1/N$ and is a function of three parameters ρ_a, ρ_b, and z. In fact, if one uses the parameter $\omega = (z-1)(\rho_a z - \rho_b - \rho_a\rho_b(z-1))/z$ defined in (2.14), one can easily check that (5.2) can be rewritten as

$$\mu = \frac{1}{N}\left(\omega - \frac{2}{3}\omega^2 + \frac{8}{45}\omega^3 + O(\omega^4)\right). \tag{5.3}$$

Up to the factor $1/N$, μ depends on the single parameter ω, defined by (2.14), (at least to order 3 in ω). The expansion of μ in powers of ρ_a and ρ_b to third order determines exactly the first three cumulants and more generally the expansion of μ to order n would give the exact expression of the first n cumulants. This can be understood by noticing the similarity between an expansion of μ in powers of $z-1$ and an expansion of μ in powers of ρ_a and ρ_b. In both cases, the hierarchy can be truncated and one can neglect all the correlations higher than the n-point function if one wishes to obtain μ at order n. This is the reason why the exact expression of the nth cumulant is a polynomial of degree n in ρ_a and ρ_b as noticed at the end of Section 2.3.

6. CONTINUOUS LIMIT

The expressions of the T_i's, $U_{i,j}$'s, $V_{i,j,k}$'s we have obtained by Mathematica to solve the hierarchy in powers of ρ_a and ρ_b are rather complicated. However they take a somewhat simpler form in the large N

limit. If one considers the connected correlation functions $u_{i,j}, v_{i,j,k},\ldots$ defined by (C.15), (C.16), their expressions become functions of the continuous variables:

$$x_1 = \frac{i}{N}, \qquad x_2 = \frac{j}{N}, \qquad x_3 = \frac{k}{N}. \tag{6.1}$$

To leading order in $1/N$ and to third order in powers of ρ_a and ρ_b, one obtains that way:

$$\begin{aligned} T_i \simeq \rho_a z[&rx_1 + (1-x_1)(1+s(-1+(1-r)^2 x_1/3 - (1-r)^2 x_1^2/6) \\ &+ s^2(1+(-23+24r-9r^2+8r^3)\, x_1/45 \\ &+ (29-42r+27r^2-14r^3)\, x_1^2/90 \\ &+ (r-1)^3 x_1^3/15 - (r-1)^3 x_1^4/60))] \end{aligned} \tag{6.2}$$

$$\begin{aligned} u_{i,j} \simeq \frac{1}{N} \rho_a^2 z^2[&x_1(1-x_2)(-(1-r)^2 + s(2(4-6r+3r^2-r^3)/3 \\ &+ (r-1)^3 x_1 - (r-1)^3 x_1^2/3 + 2(r-1)^3 x_2/3 - (r-1)^3 x_2^2/3))] \end{aligned} \tag{6.3}$$

$$v_{i,j,k} \simeq \frac{1}{N^2} \rho_a^3 z^3[-2x_1(1-2x_2)(1-x_3)] \tag{6.4}$$

where the parameters r and s are defined by

$$r = \frac{\rho_b}{\rho_a z} \tag{6.5}$$

$$s = \rho_a(z-1). \tag{6.6}$$

By examining the hierarchy (see Appendix C) for the connected functions and by assuming that the structure obtained up to third order persists to higher orders one expects that to leading order in N

$$T_i \simeq \rho_a z f(x_1) \tag{6.7}$$

$$u_{i,j} \simeq \frac{\rho_a^2 z^2}{N} g(x_1, x_2) \tag{6.8}$$

$$v_{i,j,k} \simeq \frac{\rho_a^3 z^3}{N^2} h(x_1, x_2, x_3). \tag{6.9}$$

With this scaling and as μ is of order $1/N$, one can neglect the right hand side of (C.23)–(C.25). One can even show that $g(0, x_2) = h(0, x_2, x_3) = i(0, x_2, x_3, x_4) = 0$ so that (see (C.22))

$$T_1 \simeq \frac{\alpha z}{\alpha z+\gamma} \tag{6.10}$$

$$\epsilon u_{1,i} \simeq \left(1-\frac{1}{z}\right)(1+\rho_a(z-1))(u_{1,i}-u_{0,i}) \tag{6.11}$$

$$\epsilon z_{1,i,j} \simeq \left(1-\frac{1}{z}\right)(1+\rho_a(z-1))(z_{1,i,j}-z_{0,i,j}) \tag{6.12}$$

and with these simplifications the hierarchy (C.18)–(C.31) in the large N regime becomes:

the equation for μ

$$\mu = s(1+s)\, f'(0) \tag{6.13}$$

the bulk equations (C.19)–(C.21)

$$s(1+s)\frac{d}{dx_1}\, g(0,x_2) = \frac{d^2}{dx_1^2}\, f(x_2) \tag{6.14}$$

$$s(1+s)\frac{d}{dx_1}\, h(0,x_2,x_3) = \left(\frac{d^2}{dx_1^2}+\frac{d^2}{dx_2^2}\right) g(x_2,x_3) \tag{6.15}$$

$$s(1+s)\frac{d}{dx_1}\, i(0,x_2,x_3,x_4) = \left(\frac{d^2}{dx_1^2}+\frac{d^2}{dx_2^2}+\frac{d^2}{dx_3^2}\right) h(x_2,x_3,x_4) \tag{6.16}$$

the left and right boundary equations

$$f(0) = \frac{1}{1+s}; \qquad f(1) = r \tag{6.17}$$

$$g(0,x) = g(x,1) = 0 \tag{6.18}$$

$$h(0,x,y) = h(x,y,1) = 0 \tag{6.19}$$

the equations for adjacent particles

$$\left(\frac{d}{dx_1}-\frac{d}{dx_2}\right) g(x,x) = -\left(\frac{df(x)}{dx_1}\right)^2 \tag{6.20}$$

$$\left(\frac{d}{dx_1}-\frac{d}{dx_2}\right) h(x,x,y) = -2\,\frac{df(x)}{dx_1}\frac{dg(x,y)}{dx_1} \tag{6.21}$$

$$\left(\frac{d}{dx_2}-\frac{d}{dx_3}\right) h(x,y,y) = -2\,\frac{df(y)}{dx_1}\frac{dg(x,y)}{dx_2} \tag{6.22}$$

$$\left(\frac{d}{dx_1}-\frac{d}{dx_2}\right) i(x, x, y, z) = -2\,\frac{df(x)}{dx_1}\,\frac{dh(x, y, z)}{dx_1} - 2\,\frac{dg(x, y)}{dx_1}\,\frac{dg(x, z)}{dx_1} \tag{6.23}$$

$$\left(\frac{d}{dx_2}-\frac{d}{dx_3}\right) i(x, y, y, z) = -2\,\frac{df(y)}{dx_1}\,\frac{dh(x, y, z)}{dx_2} - 2\,\frac{dg(x, y)}{dx_2}\,\frac{dg(y, z)}{dx_1} \tag{6.24}$$

$$\left(\frac{d}{dx_3}-\frac{d}{dx_4}\right) i(x, y, z, z) = -2\,\frac{df(z)}{dx_1}\,\frac{dh(x, y, z)}{dx_3} - 2\,\frac{dg(x, z)}{dx_2}\,\frac{dg(y, z)}{dx_2}. \tag{6.25}$$

One can then solve this hierarchy, up to an arbitrary order in s. To find μ at order n in s, one needs to know f at order $n-1$, g at order $n-2$ and so on. When we calculated μ at order 4 in s, we obtained

$$\begin{aligned} f(x_1) = r + \left(r - \frac{1}{s+1}\right)(x_1 - 1)\Bigg[& 1 - \omega\,\frac{x_1(x_1-2)}{6} \\ & + \omega^2\,\frac{x_1(3x_1^3 - 12x_1^2 + 28x_1 - 32)}{180} \\ & - \omega^3\,\frac{x_1(5x_1^5 - 30x_1^4 + 138x_1^3 - 352x_1^2 + 600x_1 - 576)}{5040}\Bigg] \end{aligned} \tag{6.26}$$

$$\begin{aligned} g(x_1, x_2) = \left(r - \frac{1}{s+1}\right)^2 x_1(x_2 - 1)\Bigg[& 1 - \omega\,\frac{2 - 3x_1 + x_1^2 - 2x_2 + x_2^2}{3} \\ & + \omega^2\left(\frac{x_1^4}{24} - \frac{x_1^3}{4} + \frac{x_1^2(26 - 10x_2 + 5x_2^2)}{36} - \frac{x_1(3 - 2x_2 + x_2^2)}{3}\right. \\ & \left. + \frac{56 - 80x_2 + 60x_2^2 - 20x_2^3 + 5x_2^4}{120}\right)\Bigg] \end{aligned} \tag{6.27}$$

$$\begin{aligned} h(x_1, x_2, x_3) = \left(r - \frac{1}{s+1}\right)^3 x_1(x_3 - 1)\Bigg[& 4x_2 - 2 \\ & - \omega\,\frac{\left[\begin{array}{c} 5x_2^3 - 15x_2^2 + 5x_2(x_1^2 - 3x_1 + x_3^2 - 2x_3 + 4) \\ - 3x_1^2 + 9x_1 - 2x_3^2 + 4x_3 - 6 \end{array}\right]}{3}\Bigg] \end{aligned} \tag{6.28}$$

$$i(x_1, x_2, x_3, x_4) = \left(r - \frac{1}{s+1}\right)^4 2x_1(x_4 - 1)(15x_2x_3 - 10x_2 - 5x_3 + 3) \tag{6.29}$$

and

$$\mu = \frac{1}{N}\left[\omega - \frac{\omega^2}{3} + \frac{8\omega^3}{45} - \frac{4\omega^4}{35} + O(\omega^5)\right] \tag{6.30}$$

where we give the expressions of f, g, h, i, and μ (except for a simple factor $r - \frac{1}{1+s}$) in powers of ω defined by (2.14) instead of s.

In principle all the expressions should depend on the two parameters s and r, but we observe that they only depend on the single parameter ω. This can be understood by noticing that if $f, g, h, \ldots$ solve the hierarchy (6.13)–(6.25) for a certain choice of r, s, then $Af + B$, $A^2 g$, $A^3 h, \ldots$ solve the same hierarchy for r', s' with the same value of μ if r' and s' satisfy

$$\frac{1}{1+s'} = A\frac{1}{1+s} + B$$

$$r' = Ar + B$$

$$As'(1+s') = s(1+s).$$

These three relations are compatible only when

$$s' - r's' - r's'^2 = s - rs - rs^2$$

so that when $\omega = s - rs - rs^2$ remains unchanged, one can easily transform the solution of the hierarchy leaving μ unchanged.

7. CONCLUSION

In this paper we have obtained the first four cumulants (2.17)–(2.20) of the integrated current for the symmetric simple exclusion process with open boundaries. To our surprise, the generating function of the integrated current (2.6), (2.15) depends on the densities of the reservoirs ρ_a and ρ_b and on the fugacity z, the parameter conjugated to the integrated current, through a single parameter ω defined in (2.14). It would be interesting to understand why this is so through a simple physical argument.

When $\rho_a = \rho_b = 1/2$, the fourth cumulant vanishes and we have conjectured that in this particular case, the distribution of the integrated current $Q(t)$ is Gaussian (in the range $\frac{Q(t)}{t} \sim \frac{1}{N}$). Based on this conjecture, we can predict (2.29), (2.30) the large deviation function of the current for arbitrary choices of ρ_a, ρ_b. For $\rho_a = 1$ and $\rho_b = 0$, the distribution of the integrated current we obtained is identical to the one known for one-dimensional quantum conductors in their metallic regime.[4, 5]

The similarity between these results is striking if we consider the drastic differences in the corresponding formalisms. In the quantum treatment of a diffusive conductor, the statistics of the time integrated current appears as the result of a convolution of a large number of independent binomial laws, one for each conduction channel.[5] In the limit of a large number of such channels (i.e., when the transverse dimension of the conductor is much larger than the Fermi wave length) the result of this convolution is governed by the universal distribution of eigenvalues of the transmission matrix for a single particle in the presence of quenched disorder. The exclusion effects induced by the Pauli principle only appear in the selection of the energy window in which single particle states contribute to the current. By contrast, the classical model considered here has no transverse degree of freedom, and the exclusion constraint plays a crucial role. To our knowledge, a complete understanding of the connection between the two models is still lacking. We simply conjecture that an intermediate description in terms of a Boltzmann equation with additional noise terms, as developed for instance in refs. 8 and 10 for the quantum diffusive case, may help to bridge the gap between the two classes of systems.

The first open question left at the end of the present paper is whether one could prove or disprove our conjecture for μ in Section 2.5. It would also be interesting to see the degree of universality of the results obtained here, i.e., how much they depend on the precise definition of the model. In particular it would be nice to see whether a more macroscopic approach could be used to calculate the fluctuations of the current.[21, 22] Another open question would be to know how our results would be modified by an asymmetry[18, 36, 37] in the bulk, in particular in the case of a weak asymmetry.[38]

APPENDIX A. THE GALLAVOTTI–COHEN RELATION

In this appendix we rederive, following Lebowitz and Spohn,[33] the Gallavotti–Cohen relation for a system with stochastic dynamics in contact with several reservoirs of particles.

Let us consider an irreducible Markov process for a system with a finite number of internal configurations $\mathcal{C}$. We assume that this system is in contact with a reservoir A (or several reservoirs A, B, C) and that during each infinitesimal time interval dt, there is a probability $W_q(\mathcal{C}', \mathcal{C})\, dt$ of a jump from $\mathcal{C}$ to $\mathcal{C}'$ with q particles transferred from reservoir A to the system during this jump. As the system is in general in contact with other reservoirs, these particles might later on be transferred to other reservoirs, so that $W_0(\mathcal{C}', \mathcal{C})$ allows jumps where the number of particles in the system is not conserved.

Imagine that the system is in equilibrium with reservoir A, that is the jumping rates $W_q(\mathscr{C}', \mathscr{C})$ satisfy the detailed balance condition

$$W_q(\mathscr{C}', \mathscr{C})\, P_{\text{eq}}(\mathscr{C}) = W_{-q}(\mathscr{C}, \mathscr{C}')\, P_{\text{eq}}(\mathscr{C}') \tag{A.1}$$

where $P_{\text{eq}}(\mathscr{C})$ is the steady state probability of the Markov process.

Clearly the detailed balance condition (A.1) implies that the average current of particles vanishes and that the probability of seeing any given jump is equal to the probability of its time reversal as it should for a system at equilibrium.

Now let us modify the dynamics by introducing a field E which enhances the injection of particles into the system so that $W_q(\mathscr{C}', \mathscr{C})$ is replaced by

$$e^{Eq} W_q(\mathscr{C}', \mathscr{C}). \tag{A.2}$$

This field E produces a current which of course fluctuates due to the stochastic nature of the Markov process.

Let us denote by $Q(t)$ the total number of particles transferred from reservoir A to the system during time t and $R_t(\mathscr{C}, Q)$ the probability of $Q(t)$, given that the system is in configuration $\mathscr{C}$ at time t. The evolution of $R_t(\mathscr{C}, Q)$ is clearly

$$\frac{d}{dt} R_t(\mathscr{C}, Q) = \sum_q \sum_{\mathscr{C}'} e^{Eq} [W_q(\mathscr{C}, \mathscr{C}')\, R_t(\mathscr{C}', Q-q) - W_q(\mathscr{C}', \mathscr{C})\, R_t(\mathscr{C}, Q)].$$

If one introduces the generating functions $r_t(\mathscr{C}, \lambda)$ defined by

$$r_t(\mathscr{C}, \lambda) = \sum_Q e^{\lambda Q} R_t(\mathscr{C}, Q)$$

they evolve according to

$$\frac{d}{dt} r_t(\mathscr{C}, \lambda) = \sum_q \sum_{\mathscr{C}'} \{e^{(E+\lambda)\, q} W_q(\mathscr{C}, \mathscr{C}')\, r_t(\mathscr{C}', \lambda) - e^{Eq} W_q(\mathscr{C}', \mathscr{C})\, r_t(\mathscr{C}, \lambda)\}.$$

This implies that for large t,

$$\langle e^{\lambda Q(t)} \rangle \sim e^{\mu(\lambda, E)\, t}$$

where $\mu(\lambda, E)$ is the largest eigenvalue of the matrix $M_{\lambda, E}$

$$M_{\lambda, E} = \sum_q e^{(E+\lambda)\, q} W_q(\mathscr{C}, \mathscr{C}') - \delta(\mathscr{C}, \mathscr{C}') \sum_q \sum_{\mathscr{C}''} e^{Eq} W_q(\mathscr{C}'', \mathscr{C}) \tag{A.3}$$

where $\delta(\mathscr{C}, \mathscr{C}') = 1$ if $\mathscr{C} = \mathscr{C}'$ and 0 if $\mathscr{C} \neq \mathscr{C}'$. Therefore to obtain $\mu(\lambda, E)$, one has to find either the right eigenvector $\psi_R(\mathscr{C})$ of this matrix which satisfies

$$\mu(\lambda, E)\, \psi_R(\mathscr{C}) = \sum_q \sum_{\mathscr{C}'} e^{(E+\lambda)\, q} W_q(\mathscr{C}, \mathscr{C}')\, \psi_R(\mathscr{C}') - \sum_q \sum_{\mathscr{C}'} e^{Eq} W_q(\mathscr{C}', \mathscr{C})\, \psi_R(\mathscr{C}) \tag{A.4}$$

or its left eigenvector $\psi_L(\mathscr{C})$

$$\mu(\lambda, E)\, \psi_L(\mathscr{C}) = \sum_q \sum_{\mathscr{C}'} e^{(E+\lambda)\, q} W_q(\mathscr{C}', \mathscr{C})\, \psi_L(\mathscr{C}') - \sum_q \sum_{\mathscr{C}'} e^{Eq} W_q(\mathscr{C}', \mathscr{C})\, \psi_L(\mathscr{C}). \tag{A.5}$$

Now if we use the detailed balance condition (A.1) for the first term in the r.h.s. of (A.5), we get

$$\begin{aligned}\mu(\lambda, E)\, \psi_L(\mathscr{C}) = {} & \sum_q \sum_{\mathscr{C}'} e^{(E+\lambda)\, q} W_{-q}(\mathscr{C}, \mathscr{C}')\, \frac{P_{\mathrm{eq}}(\mathscr{C}')}{P_{\mathrm{eq}}(\mathscr{C})}\, \psi_L(\mathscr{C}') \\ & - \sum_q \sum_{\mathscr{C}'} e^{Eq} W_q(\mathscr{C}', \mathscr{C})\, \psi_L(\mathscr{C}).\end{aligned} \tag{A.6}$$

This shows that $\psi_L(\mathscr{C})\, P_{\mathrm{eq}}(\mathscr{C})$ is the *right* eigenvector of the matrix $M_{-\lambda-2E,\, E}$ defined in (A.3).

So the matrices $M_{\lambda,\, E}$ and $M_{-\lambda-2E,\, E}$ have exactly the same eigenvalues. In particular this shows that

$$\mu(\lambda, E) = \mu(-\lambda - 2E, E) \tag{A.7}$$

which is the Gallavotti–Cohen relation.

In the symmetric exclusion process, as described in Section 2.1, we know[20] that detailed balance is satisfied whenever

$$\frac{\alpha}{\alpha+\gamma} = \frac{\delta}{\beta+\delta}. \tag{A.8}$$

If we fix β and δ and vary α and γ, the detailed balance condition (A.8) is no longer verified. However, one can always think of the variation of α and γ as the effect of an external field E trying to enhance the number of particles transferred from the left reservoir to site 1. If one writes

$$\alpha = \alpha' e^{E} \qquad \text{and} \qquad \gamma = \gamma' e^{-E}$$

with

$$\alpha' = \frac{\delta}{\beta}\,\gamma' = \sqrt{\frac{\alpha\gamma\delta}{\beta}}$$

and

$$e^{2E} = \frac{\alpha\beta}{\gamma\delta},$$

one sees that the system satisfies detailed balance for α', γ', β, δ. Therefore the Gallavotti–Cohen symmetry implies (2.9) for the SSEP.

APPENDIX B. THE ANALOGY WITH A MULTI-PARTICLE RUIN PROBLEM

In this appendix, we show the similarity between the equations one has to solve at each level of the hierarchy and the equations which one can write in a multi-particle ruin problem.

Consider first a single particle which diffuses on a chain of N sites with open boundary conditions. If the particle is at site i at time t, it jumps, during an infinitesimal time interval dt, to site $i+1$ with probability dt (for $1 \leqslant i \leqslant N-1$) and to site $i-1$ with probability dt (for $2 \leqslant i \leqslant N$). Moreover, a particle at site 1 is absorbed at the left boundary with probability $\alpha\, dt$ and a particle at site N is absorbed at the right boundary with probability $\beta\, dt$. In the usual ruin problem,[39] one asks the following question: what is the probability T_i that a particle starting at site i will escape at the left boundary. Clearly T_i satisfies for $2 \leqslant i \leqslant N-1$

$$T_{i+1} + T_{i-1} - 2T_i = 0$$

and at the boundaries

$$\alpha + T_2 - (1+\alpha)\, T_1 = 0$$

$$T_{N-1} - (1+\beta)\, T_N = 0.$$

These are precisely the equations (4.7)–(4.9) we had to solve in Section 4, if one takes $\gamma = \delta = 0$ and $z = 1$ (which implies that $\mu = 0$ see (4.6)).

The solution of this ruin problem is of course linear in i

$$T_i = \frac{N + \frac{1}{\beta} - i}{N + \frac{1}{\alpha} + \frac{1}{\beta} - 1}. \tag{B.1}$$

Let us now generalize the ruin problem to two particles (the generalization to more particles is straightforward). Consider two particles initially at sites $i<j$ which diffuse in the same way as in the one-particle ruin problem, except that the two particles are not allowed to occupy the same site. As time goes on, one of the two particles will escape at one of the two boundaries, then the other particle will diffuse until it also escapes.

Now we want to calculate the probability $U_{i,j}$ that both particles will escape through the left boundary. One can write down the equations satisfied by $U_{i,j}$

$$U_{i-1,j}+U_{i+1,j}+U_{i,j-1}+U_{i,j+1}-4U_{i,j}=0 \tag{B.2}$$

$$U_{i,i}+U_{i+1,i+1}=2U_{i,i+1} \tag{B.3}$$

$$(3+\alpha)\,U_{1,i}=\alpha T_i+U_{2,i}+U_{1,i+1}+U_{1,i-1} \tag{B.4}$$

$$(3+\beta)\,U_{i,N}=U_{i,N-1}+U_{i+1,N}+U_{i-1,N} \tag{B.5}$$

and they are identical to (4.12)–(4.15) when $\gamma=\delta=0$ and $z=1$ (implying that $\mu=0$). It is not obvious *a priori* that the solution of these equations is simple. However the solution turns out to be linear in i and j

$$U_{i,j}=\frac{(N+\frac{1}{\beta}-1-i)(N+\frac{1}{\beta}-j)}{(N+\frac{1}{\alpha}+\frac{1}{\beta}-1)(N+\frac{1}{\alpha}+\frac{1}{\beta}-2)}. \tag{B.6}$$

We also see in (B.1), (B.6) that the correlation between the two particles is weak

$$u_{i,j}=U_{i,j}-T_iT_j=O\left(\frac{1}{N}\right)$$

when i and j are of order N. This weak correlation,[34] which are similar to those seen in (6.8), (6.9), is however responsible for the non-Poissonian character of the fluctuations of the integrated current.

Another quantity which has a simple expression (i.e., for which the solution is linear in i and j) is the probability $U_{i,j}$ that one particle escapes at the right and the other particle escapes at the left, without specifying on which side the first particle to escape leaves (starting with two particles at positions i and j). In this case the equations to solve are again (B.2), (B.3) with boundary conditions (B.4), (B.5) replaced by

$$(3+\alpha)\,U_{1,i}=\alpha(1-T_i)+U_{2,i}+U_{1,i+1}+U_{1,i-1} \tag{B.7}$$

$$(3+\beta)\,U_{i,N}=\beta T_i+U_{i,N-1}+U_{i+1,N}+U_{i-1,N} \tag{B.8}$$

and the solution is

$$U_{i,j}=\frac{N(i+j)-2ij-2N+2i+\frac{1}{\alpha}(2N-i-j)+\frac{1}{\beta}(i+j-2)+\frac{2}{\alpha\beta}}{(N-1+\frac{1}{\alpha}+\frac{1}{\beta})(N-2+\frac{1}{\beta}+\frac{1}{\alpha})}.$$

If one asks however a slightly more precise question, namely what is the probability $U_{i,j}$ that (starting with a particle at i and a particle at j), the first particle to escape leaves at the right boundary, and then the remaining particle escapes at the left boundary, the equations to solve are still (B.2), (B.3) but with the boundary conditions (B.4), (B.5) replaced by

$$(3+\alpha)\,U_{1,i}=U_{2,i}+U_{1,i+1}+U_{1,i-1} \tag{B.9}$$

$$(3+\beta)\,U_{i,N}=\beta T_i+U_{i,N-1}+U_{i+1,N}+U_{i-1,N}. \tag{B.10}$$

These new boundary conditions make the problem much harder and one can check that the solution is no longer linear (or even polynomial) in i and j.

So the same problem (B.2), (B.3) with the boundary conditions (B.4), (B.5) or (B.7), (B.8) is easy (the solution is linear in i and j) whereas it is hard with boundary conditions (B.9), (B.10). The main reason which made possible the calculation of the cumulants in the present paper is that each time we had to solve equations of the type (B.2), (B.3), the boundary conditions were such that the solution was polynomial in the coordinates i and j.

APPENDIX C. THE HIERARCHY

C.1. The Hierarchy for the Correlation Functions

The hierarchy of Section 4 can be summarized as follows:

The equation for μ

$$\mu=\alpha(z-1)-\frac{(\alpha z+\gamma)(z-1)}{z}T_1. \tag{C.1}$$

The bulk equations

$$\mu T_i = \alpha(z-1)\, T_i - \frac{(\alpha z+\gamma)(z-1)}{z}\, U_{1,i} + T_{i-1} + T_{i+1} - 2T_i \tag{C.2}$$

$$\begin{aligned}\mu U_{i,j} = {} & \alpha(z-1)\, U_{i,j} - \frac{(\alpha z+\gamma)(z-1)}{z}\, V_{1,i,j} \\ & + U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j}\end{aligned} \tag{C.3}$$

$$\begin{aligned}\mu V_{i,j,k} = {} & \alpha(z-1)\, V_{i,j,k} - \frac{(\alpha z+\gamma)(z-1)}{z}\, W_{1,i,j,k} + V_{i-1,j,k} + V_{i+1,j,k} \\ & + V_{i,j-1,k} + V_{i,j+1,k} + V_{i,j,k-1} + V_{i,j,k+1} - 6V_{i,j,k}.\end{aligned} \tag{C.4}$$

The left boundary equations

$$\alpha(z-1)\, T_1 - \frac{(\alpha z+\gamma)(z-1)}{z}\, U_{1,1} + T_0 - T_1 = \alpha z - (\alpha z+\gamma)\, T_1 \tag{C.5}$$

$$\alpha(z-1)\, U_{1,i} - \frac{(\alpha z+\gamma)(z-1)}{z}\, V_{1,1,i} + U_{0,i} - U_{1,i} = \alpha z T_i - (\alpha z+\gamma)\, U_{1,i} \tag{C.6}$$

$$\alpha(z-1)\, V_{1,i,j} - \frac{(\alpha z+\gamma)(z-1)}{z}\, W_{1,1,i,j} + V_{0,i,j} - V_{1,i,j} = \alpha z U_{i,j} - (\alpha z+\gamma)\, V_{1,i,j}. \tag{C.7}$$

The right boundary equations

$$\delta - (\beta+\delta)\, T_N = T_{N+1} - T_N \tag{C.8}$$

$$\delta T_i - (\beta+\delta)\, U_{i,N} = U_{i,N+1} - U_{i,N} \tag{C.9}$$

$$\delta U_{i,j} - (\beta+\delta)\, V_{i,j,N} = V_{i,j,N+1} - V_{i,j,N}. \tag{C.10}$$

The equations for adjacent particles

$$U_{i,i} + U_{i+1,i+1} = 2U_{i,i+1} \tag{C.11}$$

$$V_{i,i,j} + V_{i+1,i+1,j} = 2V_{i,i+1,j} \tag{C.12}$$

$$V_{i,j,j} + V_{i,j+1,j+1} = 2V_{i,j,j+1}. \tag{C.13}$$

When one solves this hierarchy up to order 3 in ρ_a and ρ_b, one obtains

$$\mu = \frac{(\rho_a z-\rho_b)(z-1)}{zN_1} + (z-1)^2\left[\frac{(a-3a^2+2a^3+b-3b^2+2b^3)(\rho_a z-\rho_b)^2}{6z^2N_1^3(N_1-1)}\right.$$
$$-\frac{(\rho_a z-\rho_b)^2}{6z^2N_1^2(N_1-1)} + \frac{\rho_a^2 z^2+\rho_b^2}{2z^2N_1(N_1-1)} - \left.\frac{\rho_a^2z^2+\rho_a\rho_b z+\rho_b^2}{3z^2(N_1-1)}\right]$$
$$+(z-1)^3\left[\frac{(a-3a^2+2a^3+b-3b^2+2b^3)^2\,(\rho_a z-\rho_b)^3}{9z^3N_1^5(N_1-1)(N_1-2)}\right.$$
$$+\frac{(-7a+30a^2-50a^3+45a^4-18a^5-7b+30b^2-50b^3+45b^4-18b^5)(\rho_a z-\rho_b)^3}{45z^3N_1^4(N_1-1)(N_1-2)}$$
$$+\frac{[(2+15a-45a^2+30a^3+15b-45b^2+30b^3)(\rho_a^2z^2+\rho_b^2)-4\rho_a\rho_b z](\rho_a z-\rho_b)}{45z^3N_1^3(N_1-1)(N_1-2)}$$
$$-\frac{(a-3a^2+2a^3+b-3b^2+2b^3)(\rho_a^3z^3-\rho_b^3)+3(\rho_a^2z^2+\rho_b^2)(\rho_a z-\rho_b)}{9z^3N_1^2(N_1-1)(N_1-2)}$$
$$+\frac{7(\rho_a^3z^3-\rho_b^3)}{9z^3N_1(N_1-1)(N_1-2)} - \frac{(\rho_a z-\rho_b)(2\rho_a^2z^2+3\rho_a\rho_b z+2\rho_b^2)}{3z^3(N_1-1)(N_1-2)}$$
$$\left.+\frac{2(\rho_a z-\rho_b)(4\rho_a^2z^2+7\rho_a\rho_b z+4\rho_b^2)\,N_1}{45z^3(N_1-1)(N_1-2)}\right] \tag{C.14}$$

where $N_1 = N+a+b-1$, $\rho_a = \alpha/(\alpha+\gamma)$, $\rho_b = \delta/(\beta+\delta)$, $a = 1/(\alpha+\gamma)$, and $b = 1/(\beta+\delta)$.

C.2. The Hierarchy for Connected Correlation Functions

If one introduces the connected functions $u_{i,j}$, $v_{i,j,k}$,... defined by

$$U_{i,j} = T_iT_j + u_{i,j} \tag{C.15}$$

$$V_{i,j,k} = T_iT_jT_k + u_{i,j}T_k + u_{i,k}T_j + u_{j,k}T_i + v_{i,j,k} \tag{C.16}$$

$$W_{i,j,k,l} = T_iT_jT_kT_l + u_{i,j}T_kT_l + u_{i,k}T_jT_l + u_{j,k}T_iT_l + u_{i,l}T_jT_k$$
$$+u_{j,l}T_iT_k + u_{k,l}T_iT_j + u_{i,j}u_{k,l} + u_{i,k}u_{j,l} + u_{i,l}u_{j,k} + v_{i,j,k}T_l$$
$$+v_{i,j,l}T_k + v_{i,k,l}T_j + v_{j,k,l}T_i + w_{i,j,k,l} \tag{C.17}$$

the hierarchy becomes

The equation for μ (obtained by combining (C.1), (C.2), and (C.5))

$$\mu\left[1-\frac{z-1}{z}T_1\right]=\frac{z-1}{z}(T_1-T_2). \tag{C.18}$$

The bulk equations

$$\epsilon u_{1,i}=T_{i-1}+T_{i+1}-2T_i \tag{C.19}$$

$$\epsilon v_{1,i,j}=u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j} \tag{C.20}$$

$$\begin{aligned}\epsilon w_{1,i,j,k}={}&v_{i-1,j,k}+v_{i+1,j,k}+v_{i,j-1,k}\\&+v_{i,j+1,k}+v_{i,j,k-1}+v_{i,j,k+1}-6v_{i,j,k}\end{aligned} \tag{C.21}$$

where ϵ is defined by

$$\epsilon=\frac{(\alpha z+\gamma)(z-1)}{z}. \tag{C.22}$$

The left boundary equations

$$(\alpha z+\gamma)\,T_1-\alpha z+T_0-T_1=\epsilon u_{1,1}-\mu T_1 \tag{C.23}$$

$$(\alpha+\gamma)\,u_{1,i}+u_{0,i}-u_{1,i}=\epsilon v_{1,1,i}-2\mu u_{1,i} \tag{C.24}$$

$$(\alpha+\gamma)\,v_{1,i,j}+v_{0,i,j}-v_{1,i,j}=\epsilon[w_{1,1,i,j}+u_{1,i}u_{1,j}]-2\mu v_{1,i,j}. \tag{C.25}$$

The right boundary equations

$$\delta-(\beta+\delta)\,T_N=T_{N+1}-T_N \tag{C.26}$$

$$-(\beta+\delta)\,u_{i,N}=u_{i,N+1}-u_{i,N} \tag{C.27}$$

$$-(\beta+\delta)\,v_{i,j,N}=v_{i,j,N+1}-v_{i,j,N}. \tag{C.28}$$

The equations for adjacent particles

$$u_{i,i}+u_{i+1,i+1}-2u_{i,i+1}=-(T_i-T_{i+1})^2 \tag{C.29}$$

$$v_{i,i,j}+v_{i+1,i+1,j}-2v_{i,i+1,j}=-2(T_i-T_{i+1})(u_{i,j}-u_{i+1,j}) \tag{C.30}$$

$$v_{i,j,j}+v_{i,j+1,j+1}-2v_{i,j,j+1}=-2(T_j-T_{j+1})(u_{i,j}-u_{i,j+1}) \tag{C.31}$$

$$\begin{aligned}w_{i,i,j,k}+w_{i+1,i+1,j,k}-2w_{i,i+1,j,k}={}&-2(T_i-T_{i+1})(v_{i,j,k}-v_{i+1,j,k})\\&-2(u_{i,j}-u_{i+1,j})(u_{i,k}-u_{i+1,k})\end{aligned} \tag{C.32}$$

etc.

(As already discussed right after (4.15), the (unphysical) values T_0, $u_{i,i}$,... are obtained by using the explicit (polynomial) expressions of T_i, $u_{i,j}$ for $i=0$, $j=i$).

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